# Spatial Environmental Economics
# Lecture 6: Monocentric City Model

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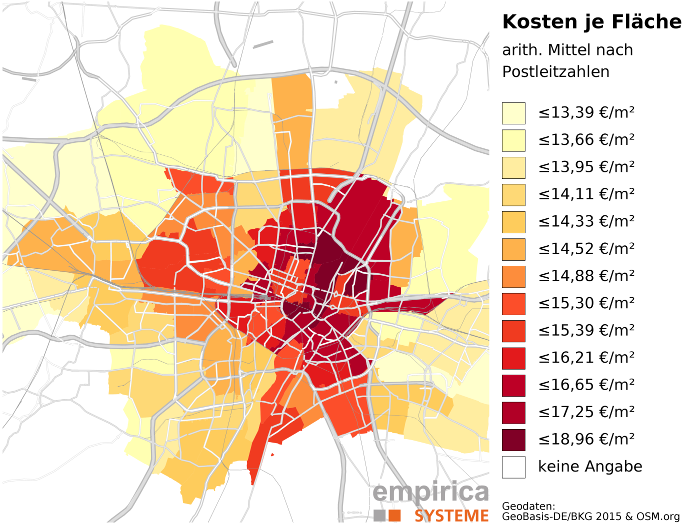
# Roadmap

# Roadmap

# Average rents by postcodes in Munich




*Note*: period 2015-08 to 2016-02. Source link: here.

# Rent gradients

- Cities almost everywhere show a decline in rent with distance to the center
- The monocentric city model predicts that pattern
  - One of the key models in urban economics
  - Also useful to motivate 'border discontinuity' hedonic valuation

# Roadmap

# General setup of the model (I/II)

- City is on a featureless plain. We assume space is a line: $S = \mathbb{R} = (-\infty, +\infty)$
  - Only a subset of $S$ will be populated
  - Locations are $i \in S$ and there is one measure of land at each $i$
- There is a landowner in each $i$ that cannot move and consumes locally
- The landowner in the city center, at $i = 0$, produces a manufactured good
  - Manufacturing good is freely traded at no cost
  - Therefore locations $|i|$ also measure distance to the center, $i > 0$ is right, $i < 0$ is left
  - Production uses labor, and pays each unit the wage $w$

# General setup of the model (II/II)

- There is a mass of identical workers/households
  - If they live in the city, choose a (residential) location $i$ and commute to work in the center
  - In the city they spend their labor income on M good, rent, and commuting costs
  - Commuting costs are $\tau_{0,i} = \tau(i)$, and increasing in distance to center $\tau'(i) > 0$
- Landowner at each $i \neq 0$ decides between renting to workers or agriculture
  - Agricultural production earns a reservation rent $\overline{R}$
- All land rent (urban and agro) is collected by 'absentee landlords' and leaves the model

# Setup of household's problem (I/II)

- Each household in the city chooses location $i$ and M consumption $c$ to maximize utility:

$$\max_{i,c} U\left(u \times c\right)$$

subject to the budget constraint

$$P\left(i\right)c + R\left(i\right) + \tau\left(i\right) = w$$

  - $u$ is the amenity of living in the city
  - Households all consume 1 unit of land and pay rent $R\left(i\right)$
  - $P\left(i\right)$ is the price of manufacturing

# Setup of household's problem (II/II)

- Notice how we can split the problem in 2 parts:

$$\max_i W_i$$

  and

$$W_i = \max_c \{U(uc) \text{ subject to } P(i)\,c + R(i) + \tau(i) = w\}$$

- $W_i$ is the indirect utility from solving the consumption choice
  - I.e. the 'well-being' of someone who lives in $i$

# Spatial Equilibrium

- To finish setting up the model, we need to describe equilibrium
- In micro the notion is typically that "markets clear" or "no excess demand"
- We want **spatial equilibrium**: broadly, 'everyone optimizes and no one wants to move'
- Two types
  1. 'Open City': all agents indifferent between all locations in the city and their outside option
  2. 'Closed City': all agents indifferent between all locations in the city, but they are not allowed to move away
- Spatial equilibrium is sometimes called 'free mobility'. That is, people move for tiny gains
- We will assume an **open city**, and that living outside the city earns a reservation utility $\overline{W}$

# Solving the model: rent gradient (I/III)

- Denote by $c^*(i)$ the **optimal consumption**, and so $W_i = U(uc^*(i))$
- **Spatial equilibrium** means that people will be indifferent between locations in the city

$$W_i = W_j \text{ for any populated } i, j \in S$$

  - Equivalently: $W_i = W$ for all populated $i$
  - Why? If not, people would want to move, so it's not an equilibrium
- **Open city** assumption means they will also be indifferent between living in the city or outside, so $W = \overline{W}$ or

$$W_i = \overline{W} \text{ for any } i \in S$$

# Solving the model: rent gradient (II/III)

- Putting all together:

$$\overline{W} = U\left(uc^*\left(i\right)\right)$$

- From that condition, we can solve for $c^*\left(i\right)$

$$c^*\left(i\right) = \frac{1}{u} U^{-1}\left(\overline{W}\right)$$

  - Nothing on the right-hand side depends on $i$!
  - So $c^*\left(i\right) = c^*$ (i.e. consumption equalized in space)
- Replacing into the budget constraint: $P\left(i\right) c^* + R\left(i\right) + \tau\left(i\right) = w$
- Since **manufacturing is freely traded**, any price differences should be arbitraged away
  - That means $P\left(i\right) = P$
  - Choose $P = 1$ as the numeraire price

# Solving the model: rent gradient (III/III)

- Replacing $P(i) = P = 1$ into the equilibrium budget constraint, we obtain

$$R(i) + \tau(i) = w - c^*$$

  - That is, with wages and consumption fixed for all households, commuting costs and land rent must vary in such a way that they always sum to a constant
- Assuming zero commuting costs in the center $(\tau(0) = 0)$ this already tells us

$$R(0) \quad = \quad w - c^*$$

  and that rents decrease with distance to the center

$$R(i) = w - c^* - \tau(i) < R(0) \;\text{ for all } i \neq 0$$

# Solving the model: city edge

- We still need to determine which locations are populated and which not
  - Define locations $f$ as the edge of the city, so that populated locations are $i \in (-f, f)$
- **Optimization by landowners** means that they will rent to workers whenever $R(i) \geq \overline{R}$
- Since $f$ is the most remote populated location, it must have the lowest rent, and so

$$R(f) = \overline{R}$$

  - I.e., the landowners at the edge will be indifferent between renting to workers and agro use
- We can once again replace into the equilibrium budget constraint to find the city edge $f$

$$\tau(f) = w - c^* - \overline{R}$$

- Since every household uses 1 unit land, the number of people in the city is $L = 2f$

# Summary of the monocentric city model

- Exogenous: $w$, $\tau(i)$, $\overline{R}$, $\overline{W}$, $u$
- Endogenous: $R(i)$,$f$ (and implied $L$)
- Equilibrium conditions:

$$
\begin{aligned}
R(i) &= \begin{cases} w - c^* - \tau(i) & \text{if } i \in [-f, f] \\ \overline{R} & \text{if else} \end{cases} \\
\tau(f) &= w - c^* - \overline{R} \\
c^* &= \frac{1}{u} U^{-1}\left(\overline{W}\right)
\end{aligned}
$$

and city size is $L = 2f$

# Roadmap

# Simplifying assumptions

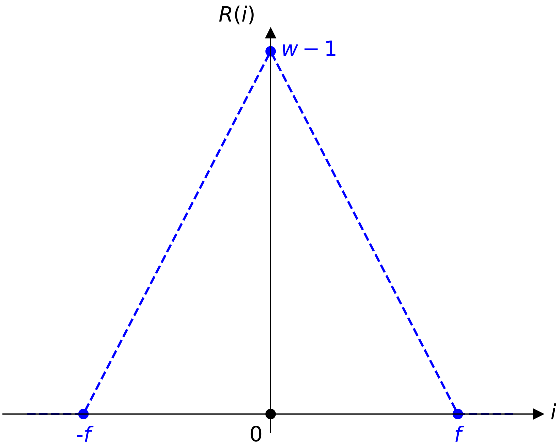
- Assume that
  - $\overline{R} = \overline{W} = 0$
  - $\tau(i) = 2t\,|i|$
  - $U(uc) = \ln(uc)$
- Then the equilibrium conditions depend only on $t$, $w$, and $u$:

$$
\begin{aligned}
R(i) &= \begin{cases} w - u^{-1} - 2t\,|i| & \text{if } i \in [-f, f] \\ 0 & \text{if else} \end{cases} \\
f &= \frac{w - u^{-1}}{2t}
\end{aligned}
$$

and city size is $L = 2f$

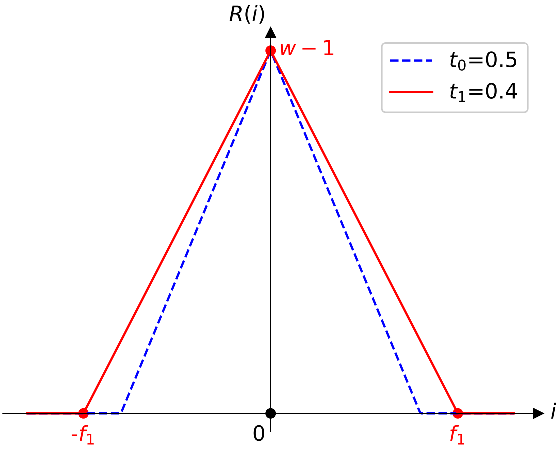
# Plot of the rent gradient

Baseline with $u = 1$, $t = 0.5$ and $w = 10$



# What happens if we reduce commuting costs? Plot

Reduce from $t_0 = 0.5$ to $t_1 = 0.4$




- Rent gradient (slope) flattens, city gets larger

# What happens if we reduce commuting costs? Analytically

- The rent gradient flattens: for $i > 0$, $R'(i) = -2t$, so

$$\frac{\partial R'(i)}{-\partial t} = 2 > 0$$

  - Land rent capitalizes the cost/value of commuting

- City extent grows $\implies$ population increases (land per person is fixed at 1)

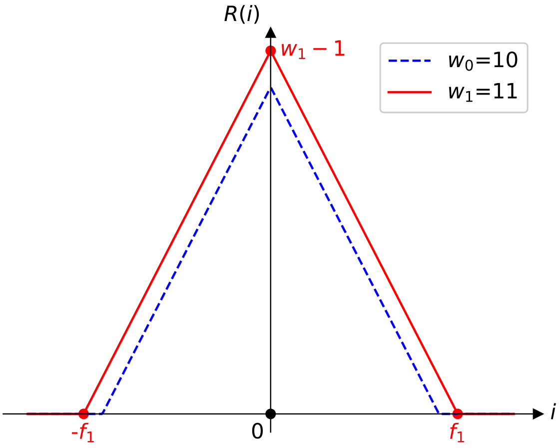
$$\frac{\partial f}{-\partial t} = \frac{w - u^{-1}}{2t^2} > 0$$

  so $\frac{\partial L}{-\partial t} = 2\frac{\partial f}{-\partial t} > 0$

- Utility and consumption of households stay the same (only a function of $u$ and $\overline{W}$)
- Notice that all of the benefit of a reduction in transportation costs either gets used up with more commuting, or is collected by absentee landlords

# What happens if the city becomes more productive? Plot

Increase from $w_0 = 10$ to $w_1 = 11$




- Rent gradient (slope) stays the same, city gets larger

# What happens if the city becomes more productive? Analytically

- The rent gradient stays the same: for $i > 0$, $R'(i) = -2t$ is not a function of $w$
- City extent grows $\implies$ population increases (land per person is fixed at 1)

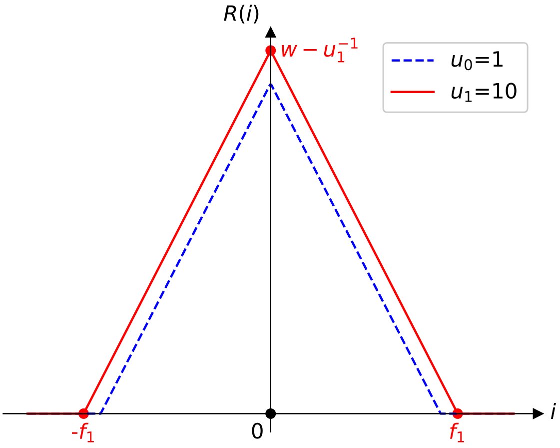
$$\frac{\partial f}{\partial w} = \frac{1}{2t} > 0$$

so $\frac{\partial L}{\partial w} > 0$

- Utility and consumption of households stay the same (only a function of $u$ and $\overline{W}$)
- Rent increases by the same amount that wage increases: $\frac{\partial R(i)}{\partial w} = 1$
- Notice that all of the benefit of a productivity improvement either gets used up with more commuting, or is collected by absentee landlords

# What happens if amenities improve? Plot

Increase from $u_0 = 1$ to $u_1 = 10$




- Rent gradient (slope) stays the same, city gets larger

# What happens if amenities improve? Analytically

- The rent gradient stays the same: for $i > 0$, $R'(i) = -2t$ is not a function of $u$
- City extent grows $\implies$ population increases (land per person is fixed at 1)

$$\frac{\partial f}{\partial u} = \frac{u^2}{2t} > 0$$

so $\frac{\partial L}{\partial w} > 0$

- Utility and consumption of households fall!

$$\frac{\partial c^*}{\partial u} = -u^{-2} < 0$$

- Rent increases by the same amount that consumption falls: $\frac{\partial R(i)}{\partial u} = u^{-2}$
- Implies that 'nicer' cities should be bigger and more expensive than less nice ones.

  Another way of saying this: land rents capitalize amenities

# Roadmap

1. Motivation
2. Model
3. Model Predictions
4. **Application: Border Discontinuity**
5. Model: Closed City
6. Model Predictions: Open vs. Closed
7. Conclusion

# Valuing amenities

- The model we just discuss implies that cities with better amenities should be bigger and more expensive
  - This is a widely used insight for valuing place specific attributes/policies
  - But notice that our model has only 1 city. The Roback model in next lecture will allow more
- We can apply the same insight within a city if we allow amenities to vary within the city
  - We can have a discontinuous rent gradient if amenities vary discontinuously
  - In this case, spatial equilibrium requires that rent vary discontinuously in order to equalize utility across locations
  - This intuition motivates the '**border discontinuity design**' for empirical work

# Model with varying amenities

- Re-write our open-city monocentric model with varying amenities $u_i$
- The equilibrium conditions are now

$$
\begin{aligned}
R(i) &= \begin{cases} w - c_i^* - \tau(i) & \text{if } i \in [-f, f] \\ \overline{R} & \text{if else} \end{cases} \\
\tau(f) &= w - c_f^* - \overline{R} \\
c_i^* &= \frac{1}{u_i} U^{-1}\left(\overline{W}\right)
\end{aligned}
$$

# Model with varying amenities: example (I/II)

- + same simplifying assumptions as before on $\overline{R}$, $\overline{W}$, $\tau(i)$ and $U(.)$:

$$R(i) = w - u_i^{-1} - 2t\,|i| \text{ if } i \in [-f, f],$$

$R(i) = 0$ if $i \notin [-f, f]$, and $f = \frac{w - u_f^{-1}}{2t}$

- Suppose only places at distance $b$ from the center have access to running water, so

$$u_i = \begin{cases} u > 1 & \text{if } i \in [-b, b] \\ 1 & \text{if else} \end{cases}$$

  - Assume we are in the interesting case when $b < f$
  - Note: $u - 1$ measures the value of water access

# Model with varying amenities: example (II/II)

- With that assumption on $u_i$, the equilibrium conditions become:

$$
\begin{aligned}
R(i) &= \begin{cases} 0 & \text{if } i < -f \\ w - 1 - 2t\,|i| & \text{if } i \in [-f, b) \\ w - u^{-1} - 2t\,|i| & \text{if } i \in [-b, b] \\ w - 1 - 2t\,|i| & \text{if } i \in (b, f] \\ 0 & \text{if } i > f \end{cases} \\
f &= \frac{w-1}{2t}
\end{aligned}
$$

# Border discontinuity

- If we are very (infinitesimally) close to the border $b$ from the left the amenity is $u$:

$$\lim_{i \to b^-} R(i) = w - u^{-1} - 2tb$$

- If we are very (infinitesimally) close to the border $b$ from the right the amenity is 1:

$$\lim_{i \to b^+} R(i) = w - 1 - 2tb$$

- Then if we compare rents to the left and right of the border:

$$\lim_{i \to b^-} R(i) - \lim_{i \to b^+} R(i) = u - 1$$

  - We recover the value of the amenity! Only knowing rents and the location of the border
  - Note: commuting costs wash out because they are continuous over $b$

# Application: school quality

- Black (1999) uses the idea of border discontinuity to examine the value of school quality
- Setting: three counties in Massachussets between 1993 and 1995
  - School attendance zone boundaries
  - Home prices from real estate transaction data
  - Average test scores as a measure of school quality
- Compares transactions within a few hundred meters of a school attendance zone boundary
- If (i) real estate markets are in spatial equilibrium and (ii) other determinants of home prices vary continuously over school zone boundaries

  $\Longrightarrow$ price differences reflect the value of improving test scores

# Roadmap

# Closed city

- We now solve the exact same model but assuming a **closed city**
- The size of the city is fixed at $\overline{L}$ (exogenous)
- Spatial equilibrium still means $W_i = W$ inside the city
- But now the utility level $W$ is endogenous

# Solving the model: rent gradient

- Following same steps as with open city, spatial equilibrium + consumer optimization imply

$$c^{*}(i) = c^{*} = \frac{1}{u} U^{-1}(W)$$

  - Note: now a function of $W$, not $\overline{W}$

- The equilibrium budget constraint ($+\ P(i) = P = 1$) gives us the rent gradient

$$R(i) = w - c^{*}(W) - \tau(i)$$

for every populated $i$

# Solving the model: city edge

- Following the same steps as with open city: $R(f) = \overline{R}$
- $+$ equilibrium budget constraint: $\tau(f) = w - c^* - \overline{R}$
- Moreover, since everyone consumes 1 unit of land the size of the city is still $\overline{L} = 2f$ so

$$f = \overline{L}/2$$

- And we can use this to solve for equilibrium consumption from the budget constraint:

$$c^* = w - \tau\left(\overline{L}/2\right) - \overline{R}$$

# Summary of the monocentric closed-city model

- Exogenous: $w$, $\tau(i)$, $\overline{R}$, $\overline{L}$, $u$
- Endogenous: $R(i)$,$f$ (and implied $W$)
- Equilibrium conditions:

$$
\begin{aligned}
R(i) &= \begin{cases} w - c^* - \tau(i) & \text{if } i \in [-f, f] \\ \overline{R} & \text{if else} \end{cases} \\
f &= \overline{L}/2 \\
c^* &= w - \tau(f) - \overline{R}
\end{aligned}
$$

and city indirect utility is $W = U(uc^*)$

# Open-city vs. closed-city spatial equilibrium

- Differ in what stays fixed (exogenous) and what adjusts (endogenous)
- With open city equilibrium, the reservation utility ($\bar{W}$) is exogenous, and the population of the city ($L$) adjusts until the marginal person pays just the agricultural rent
- With closed city equilibrium, population is fixed ($\bar{L}$) and utility ($W$) adjusts so the person at zero pays just enough that the person at the edge doesn't want to outbid him for the central spot
- Open and Closed city are special cases where migration is free or infinitely expensive
  - Reality is going to lie in between (need a different model!)

# Roadmap

# Simplifying assumptions

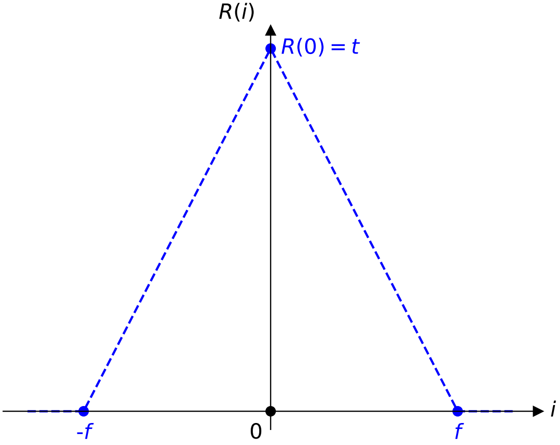
- Assume that
  - $\overline{R} = 0$, $\overline{L} = 1$
  - $\tau(i) = 2t\,|i|$
  - $U(uc) = \ln(uc)$
- Then the equilibrium conditions depend only on $t$, $w$, and $u$:

$$
\begin{aligned}
R(i) &= \begin{cases} t\,(1 - 2\,|i|) & \text{if } i \in [-f, f] \\ 0 & \text{if else} \end{cases} \\
f &= 1/2 \\
c^* &= w - t
\end{aligned}
$$

and city indirect utility is $W = \ln(u \times c^*)$

# Plot of the rent gradient

Baseline with $t = 0.5$



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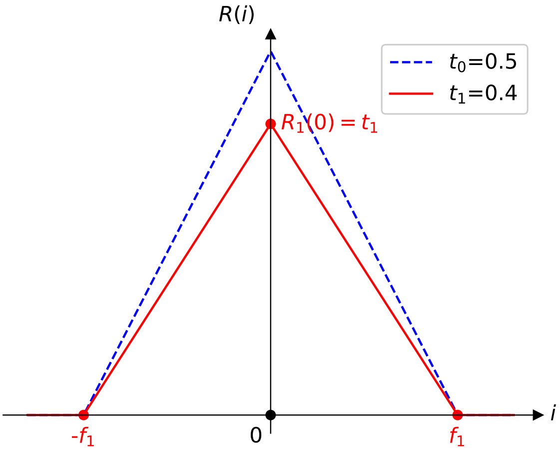
Reduce from $t_0 = 0.5$ to $t_1 = 0.4$




- Rent gradient (slope) flattens

# What happens if we reduce commuting costs? Analytically

- The rent gradient flattens: for $i > 0$, $R'(i) = -2t$, so

$$\frac{\partial R'(i)}{-\partial t} = 2 > 0$$

  - Land rent capitalizes the cost/value of commuting
- City size stays the same (follows from closed city + fixed land consumption)
- Utility and consumption of households increase: $\frac{\partial c^*}{-\partial t} = 1$
- Rents fall: $\frac{\partial c^*}{\partial t} = 1 - 2|i| > 0$ for $i \in (-f, f)$
- With a closed city, all of the benefit of the improved transportation costs are captured by residents. Landlords are worse off

# Role of amenities and manufacturing productivity

- With an open city, changes in wages $w$ and amenities $u$ only affect households
- Interesting parallel with tax incidence:
  - With open city, supply of people perfectly elastic. All changes fall on landowners, good or bad
  - With closed city, supply of people perfectly inelastic, so changes fall on households
- This highlights the importance of understanding migration responses to local conditions
  - Policy evaluation can yield very different conclusions
  - Key insight for understanding the incidence of spatial shocks such as climate change

# Roadmap

# Conclusion

- We have a model that assumes:
  - Transportation is costly
  - Everyone wants to work in the center
  - People arrange themselves so that no one wants to move, i.e., spatial equilibrium
- With just these assumptions, we get a downward sloping rent gradient
- Limitations
  - Why are people in the center? This is a central assumption
  - Why only one center? Many (most) cities are not monocentric
  - What if people/households are not identical?
- Next: build on these ideas to develop richer models that can overcome those limitations

# Appendix

# References I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