Spatial Environmental Economics
Lecture 7: Roback Model

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Roadmap

Roadmap

1. A Basic Roback Model
2. Roback with housing
3. Valuing Amenities
4. Application: Value of Climate
5. Conclusion

Setup (I/II)

Simplified version of Jennifer Roback's landmark paper (Roback, 1982)

- **Space**:
  - An arbitrary number of discrete locations $i \in S = \{1, 2, 3, ...\}$ (think: cities in a country)
  - No barriers to moving goods or people across space (no role for τ_{ij})
- Agents: households, firms and landowners
- **Landowners** consume outside the model ("absentee landlords")
  - Alternative: re-distribute rent profits lump-sum to households
- **Firms**
  - A homogeneous good is produced at all locations
  - Perfect competition
  - CRS (linear) technology using labor with productivity A_i ($Y_i = A_i L_i$)

Setup (II/II)

- **Households** (who are also workers and consumers)
  - There is a total of $\bar{L}$ (exogenous) identical households
  - Welfare $W_i = C_i u_i$ where C_i = consumption, u_i = amenity
  - Budget constraint $P_i C_i = w_i$
- Externalities:
  - **Agglomeration on labor productivity** $A_i = \bar{A}_i L_i^{\alpha}$, $\alpha \geq 0$
  - **(Negative) Externalities on amenity** $u_i = \bar{u}_i L_i^{-\beta}$, $\beta > 0$
- Assume $\beta > \alpha$
  - So that model is 'well-behaved': welfare decreases with population
  - If $\beta < \alpha$, possibility of 'black hole'! All population may end up in one location

Equilibrium

- Households problem: $W_i = w_i u_i / P_i$ (i.e. $C_i = w_i / P_i$)
- Producer problem (perfect competition): $A_i P_i = w_i$
- No trade costs require no arbitrage: $P_i = P$. Normalize $P = 1$
- Spatial equilibrium:

$$\begin{aligned} W_i &= W \text{ for all } i \\ \sum_j L_j &= \bar{L} \end{aligned}$$

  - i.e. welfare equalization and everyone lives somewhere

Equilibrium solution (I/II)

- Combining producer problem + consumer problem + no arbitrage:

$$W_i = \bar{A}_i \bar{u}_i L_i^{\alpha-\beta} \tag{1}$$

- Spatial equilibrium:

$$W_i = W \tag{2}$$

$$\sum_j L_j = \bar{L} \tag{3}$$

Equilibrium solution (II/II)

Replacing (1) into (2):

$$L_i = \left(\frac{\bar{A}_i \bar{u}_i}{W} \right)^{\frac{1}{\beta-\alpha}}$$

replace into (3) to solve for W:

$$W = \left[\frac{\sum_j \left(\bar{A}_j \bar{u}_j \right)^{\frac{1}{\beta-\alpha}}}{\bar{L}} \right]^{\beta-\alpha}$$

Plugging in W into L_i:

$$L_i = \frac{\left(\bar{A}_i \bar{u}_i \right)^{\frac{1}{\beta-\alpha}}}{\sum_j \left(\bar{A}_j \bar{u}_j \right)^{\frac{1}{\beta-\alpha}}} \bar{L}$$

Equilibrium conditions

- Therefore the equilibrium is

$$L_i = \frac{\left(\bar{A}_i \bar{u}_i\right)^{\frac{1}{\beta-\alpha}}}{\sum_j \left(\bar{A}_j \bar{u}_j\right)^{\frac{1}{\beta-\alpha}}} \bar{L}$$

$$W = \left[\sum_j \left(\bar{A}_j \bar{u}_j\right)^{\frac{1}{\beta-\alpha}}\right]^{\beta-\alpha} \bar{L}^{-(\beta-\alpha)}$$

- These equations map exogenous variables $(\bar{A}_i, \bar{u}_i, \bar{L})$ to endogenous variables (W, L_i)

Implication 1: population reflects fundamentals

$$\frac{L_i}{\bar{L}} = \frac{\left(\bar{A}_i \bar{u}_i\right)^{\frac{1}{\beta-\alpha}}}{\sum_j \left(\bar{A}_j \bar{u}_j\right)^{\frac{1}{\beta-\alpha}}}$$

- More people in i if, all else equal:
  - It is more productive ($\iff$ $\bar{A}_i$ higher)
  - It has better amenities ($\iff$ $\bar{u}_i$ higher)
- Another way of saying this is that L_i depends on i's relative attractiveness

Implication 2: wages reflect compensating differentials

- Consumer optimization + welfare equalization + amenity spillover:

$$w_i = \frac{L_i^{\beta} W}{\bar{u}_i}$$

  - So, all else equal, wages decreasing in amenities
  - But it also depends on the equilibrium allocation of L_i
  - ...and thus the correlation of $\bar{A}_i$ and $\bar{u}_i$ in space

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Setup

- We want to include rents (or home prices) in the model
  - Building toward an application where housing prices are used to value amenities
- Exactly the same assumptions as before, except for
  - An extension: households consume housing space too, and there is an exogenous supply $\bar{H}_i$
  - A simplification (not required): no agglomeration externalities ($\alpha = 0$)
- Assume the households have Cobb-Douglas preferences given by

$$U\left(\bar{u}_i, c, h\right) = \bar{u}_i \left(\frac{c}{1-\beta}\right)^{1-\beta} \left(\frac{h}{\beta}\right)^{\beta}$$

and the budget constraint is $P_i c + R_i h = w_i$

Aside: what is an amenity u_j?

Two notions of amenities in spatial economics (Dingel, 2023):

1. Place-specific **services that are not explicitly transacted**, so do not appear in the budget constraint
   - "Unlike for other goods, increments to amenities can be gained solely through a change in location" (Diamond and Tolley, 1982)
   - In the case of "consumption" or "retail" amenities, "it is the option to buy the good at a given price, and not the good itself, which is location-specific and thus an amenity"
2. Place-specific **residuals** because the researcher lacks relevant price or expenditure data
   - Residuals also reflect *unobserved* price, variety, and quality variation

Equilibrium

- Households problem *[NEW]*: indirect utility is

$$W_i = \frac{\bar{u}_i w_i}{P_i^{1-\beta} R_i^{\beta}}$$

- Producer problem (perfect competition): $A_i P_i = w_i$
- No trade costs require no arbitrage: $P_i = P$. Normalize $P = 1$
- Housing market clearing *[NEW]*:

$$\bar{H}_i = \beta \frac{w_i L_i}{R_i}$$

- Spatial equilibrium:

$$W_i = W \text{ for all } i \text{ and } \sum_j L_j = \bar{L}$$

Equilibrium solution (I/II)

- Combining producer problem + consumer problem + no arbitrage:

$$W_i = \bar{u}_i \bar{A}_i R_i^{-\beta} \tag{4}$$

- Housing market clearing + producer problem:

$$R_i \bar{H}_i = \beta \bar{A}_i L_i \tag{5}$$

- Spatial equilibrium:

$$W_i = W \tag{6}$$

$$\sum_j L_j = \bar{L} \tag{7}$$

Equilibrium solution (II/II)

Replacing (6) and (5) into (4):

$$L_i = \left(\frac{\bar{u}_i \bar{A}_i^{1-\beta} \bar{H}_i^{\beta}}{\beta^{\beta} W} \right)^{\frac{1}{\beta}}$$

replace into (7) to solve for W:

$$W = \left[\frac{\sum_j \left(\bar{u}_j \bar{A}_j^{1-\beta} \bar{H}_j^{\beta} \right)^{\frac{1}{\beta}}}{\beta \bar{L}} \right]^{\beta}$$

Plugging in W into L_i:

$$L_i = \frac{\left(\bar{u}_i \bar{A}_i^{1-\beta} \bar{H}_i^{\beta} \right)^{\frac{1}{\beta}}}{\sum_j \left(\bar{u}_j \bar{A}_j^{1-\beta} \bar{H}_j^{\beta} \right)^{\frac{1}{\beta}}} \bar{L}$$

Equilibrium conditions

- Therefore the equilibrium is

$$L_i = \frac{\left(\bar{u}_i \bar{A}_i^{1-\beta} \bar{H}_i^{\beta}\right)^{\frac{1}{\beta}}}{\sum_j \left(\bar{u}_j \bar{A}_j^{1-\beta} \bar{H}_j^{\beta}\right)^{\frac{1}{\beta}}} \bar{L}$$

$$W = \left[\sum_j \left(\bar{u}_j \bar{A}_j^{1-\beta} \bar{H}_j^{\beta}\right)^{\frac{1}{\beta}}\right]^{\beta} \left(\beta \bar{L}\right)^{-\beta}$$

- These equations map exogenous variables $(\bar{A}_i, \bar{u}_i, \bar{H}_i, \bar{L})$ to endogenous variables (W, L_i)

Model implications (I/II)

- Revisit the implications of the Roback model by comparing any 2 locations $i \neq k$
- **Population reflects fundamentals**. From spatial equilibrium + housing market clearing:

$$\frac{L_i}{L_k} = \left(\frac{w_i}{w_k}\right)^{\frac{1-\beta}{\beta}} \left(\frac{\bar{u}_i}{\bar{u}_k}\right)^{\frac{1}{\beta}} \frac{\bar{H}_i}{\bar{H}_k}$$

  - At equal wages, population is higher where fundamentals (amenities and housing) are better
  - It is in this sense that fundamentals can be "place-specific residuals", they explain the differences in population densities that we cannot explain with differences in wages

Model implications (II/II)

- **Wages reflect compensating differentials**. From the spatial equilibrium condition:

$$\frac{w_i}{w_k} = \frac{\bar{u}_k}{\bar{u}_i}\left(\frac{R_i}{R_k}\right)^{\beta}$$

  - Given housing costs, wages reflect amenities

- Moving rents to the left we can make a slightly more general point:

$$\frac{R_i^{\beta}/w_i}{R_k^{\beta}/w_k} = \frac{\bar{u}_i}{\bar{u}_k}$$

  - "Cost of living" (≈housing cost as share of income) differences reflect amenity differences
  - Key intuition of what follows

Roadmap

Roback theorem: preliminaries

- This model gives us a way to think about the value of place-specific attributes ($\bar{u}_i$)
  - Handy because these attributes often don't have an explicit market price
- Goal: derive an implicit 'price' of amenities from the indirect utility:

$$p^u \equiv \frac{\frac{\partial W}{\partial \bar{u}}}{\frac{\partial W}{\partial w}}$$

  - The 'implicit price' is the ratio of the marginal utility of $\bar{u}$ to the marginal utility of income
  - Commonly referred to as the marginal willingness to pay (MWTP) for $\bar{u}$
  - It measures the amount of income one is willing to forgo in order to increase the amenity

Roback theorem: derivation (I/II)

1. Combining spatial equilibrium + consumer optimization + no arbitrage we have

$$W = W_i\left(w_i, R_i, \bar{u}_i\right)$$

2. Total differentiation:

$$0 = \frac{\partial W_i}{\partial w_i} dw_i + \frac{\partial W_i}{\partial \bar{u}_i} d\bar{u}_i + \frac{\partial W_i}{\partial R_i} dR_i$$

   - ≈'what small changes in the arguments of W_i would still satisfy the equilibrium equation'

3. Solve for p^u:

$$\underbrace{\frac{\partial W_i / \partial \bar{u}_i}{\partial W_i / \partial w_i}}_{p^u} = -\frac{\partial W_i / \partial R_i}{\partial W_i / \partial w_i} \frac{dR_i}{d\bar{u}_i} - \frac{dw_i}{d\bar{u}_i}$$

Roback theorem: derivation (II/II)

4. Use $W_i = w_i \bar{u}_i R_i^{-\beta}$ to obtain

$$\begin{aligned} \frac{\partial W_i}{\partial R_i} &= -\frac{\beta}{R_i} \frac{w_i \bar{u}_i}{R_i^{\beta}} \\ \frac{\partial W_i}{\partial w_i} &= \frac{\bar{u}_i}{R_i^{\beta}} \end{aligned}$$

5. Replace into the expression for p^u to obtain

$$p_i^u = \beta \frac{w_i}{R_i} \frac{dR_i}{d\bar{u}_i} - \frac{dw_i}{d\bar{u}_i}$$

Roback theorem: statement and interpretation

- Roback theorem: the 'price' (MWTP) for amenity $\bar{u}$ is given by

$$p_i^u = \beta \frac{w_i}{R_i} \frac{dR_i}{d\bar{u}_i} - \frac{dw_i}{d\bar{u}_i}$$

- Intuition: the value of a marginal increase in the amenity $\bar{u}$ reflects
  - the total change in housing expenditure required to obtain it (first term)
  - the change in income (second term)
- Usefulness: we can calculate the value of an amenity using readily available data
  - $\beta w_i / R_i$ can be measured from data on housing expenditures
  - $dR_i / d\bar{u}_i$ can be estimated from a regression of house prices on amenities
  - $dw_i / d\bar{u}_i$ can be estimated from a regression of income or wages on amenities

Roadmap

Albouy et al. (2016)

"Climate Amenities, Climate Change, and American Quality of Life"

- Questions
  - What is the value of climate amenities in the United States?
  - What will be the welfare impact of climate change?
- Approach
  - Measure WTP (implicit prices) with regressions based on our Roback model, with

$$\ln \bar{u}_i = \delta T_i + \xi_i$$

    - T_i: average temperature in i (or other observed climate variables)
    - ξ_i: amenity observed by households but not by us
  - Estimate with cross-sectional data

Deriving an estimating equation for the WTP for climate

- Start from the spatial equilibrium condition: $R_i^\beta / w_i = \bar{u}_i / W$ implies

$$\ln \left(R_i^\beta / w_i \right) = \ln \bar{u}_i - \ln W$$

- Replacing $\bar{u}_i = \exp(\delta T_i + \xi_i)$:

$$\underbrace{\ln \left(R_i^\beta / w_i \right)}_{\text{data}} = \delta T_i - \underbrace{\ln W}_{\text{constant}} + \xi_i$$

- So our estimand (=parameter that we want to estimate) is

$$\delta = \frac{\partial \ln \left(R_i^\beta / w_i \right)}{\partial T_i}$$

Link to the Roback Theorem

- We can show that $\delta = p_i^T / w_i$ (i.e. WTP for an extra degree as a fraction of income)
- Start from the Roback theorem:

$$\frac{p_i^T}{w_i} = \beta \frac{dR_i / R_i}{dT_i} - \frac{dw_i / w_i}{dT_i}$$

- And then use properties of derivatives and logarithms:

$$\beta \frac{dR_i / R_i}{dT_i} - \frac{dw_i / w_i}{dT_i} = \beta \frac{d \ln R_i}{dT_i} - \frac{d \ln w_i}{dT_i} = \frac{d}{dT_i} (\beta \ln R_i - \ln w_i) = \frac{d \ln \left(R_i^\beta / w_i \right)}{dT_i} = \delta$$

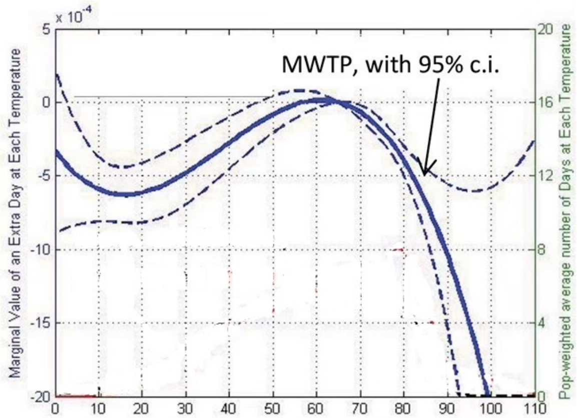
Data

- Spatial resolution: 2,057 Public Use Microdata Areas (PUMAs), populations 100k-300k
- 2000 U.S. Census: income, housing costs, demographics
- Daily average temperature data over 1970–1999
- Predicted climate data: Community Climate System Model (CCSM) used in IPCC 2007, A2 business-as-usual scenario

Results: WTP for average daily temperature

Cubic spline: $\ln \bar{u}_i = \sum_{s=1}^{5} \delta_s S_s(T_i) + \xi_i$ where $S_s(t)$ is standard basis function

- WTP is maximized at 18 °C (65 °F)
- Departs nonlinearly away from 18 °C
- Greater WTP to avoid heat than to avoid cold (steeper on the right)



Results: discussion

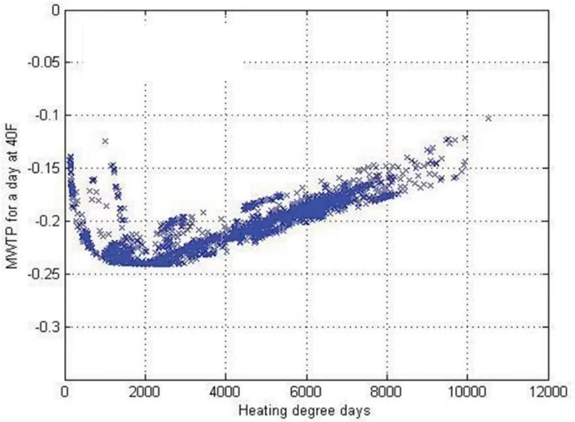
- Could the results be driven by differences between households?
  - Households with high WTP to avoid heat may select into cooler areas (sorting)
  - Different air conditioner penetration (adaptation)
- Paper addresses this by estimating location-specific MWTP using local-linear regression

$$\ln \bar{u}_i = \delta^i T_i + \xi_i$$

  - Idea: estimate δ^i weighing data for locations that have a similar temperature as i
  - Allows WTP to avoid heat (cold) to be different in hot (cold) locations

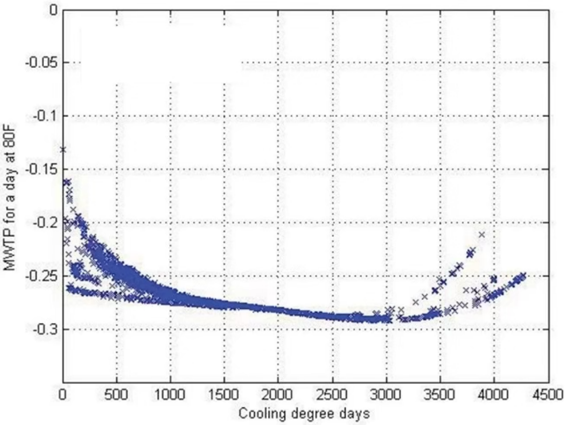
Results: WTP for a **cold** (4.4 °C) day across locations

- Upward slope
- Households with the most negative MWTP for cold weather tend to be located in areas with the fewest HDDs
- Consistent with sorting and with adaptation to cold climates
- (Note: large confidence intervals close to zero HDD, not shown here)

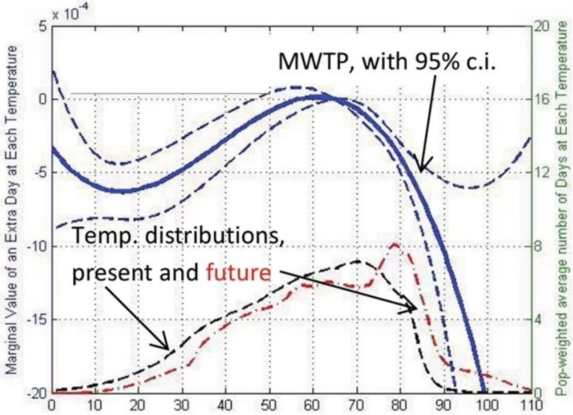


Results: WTP for a **hot** (26.7 °C) day across locations

- Weak downward slope
- Consistent with residents of hot areas being more heat averse
- Suggests limited ability to adapt to heat, relative to cold
- Heterogeneity in MWTP is smaller than that for cold



Results: climate change



Results: climate change (continued)

- They find a WTP of 1% to 4% of income to avoid CC expected by 2100
- Welfare losses driven by:
  - WTP to avoid heat > WTP to avoid cold, so milder winters in some regions do not compensate for hotter summers in other regions
  - Much of the country lose many days with moderate temperatures, which are highly valued
- Discussion
  - Holding technology and preferences constant
  - MWTP identified at their chosen climate, cannot identify WTP for a *large* change in climate
  - Migration? (although it has a limited scope because few areas actually improve)

Results: climate change (continued)

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Conclusion

- The **Roback model** is a workhorse model in urban and spatial economics
  - It provides a method to use data to estimate the value of an amenity
  - The required data is typically easily observable: information on wages, rents, and amenities
  - Good starting point to think about public funding of, e.g., crime reduction or better parks
- Example of a **revealed preference method** for valuing environmental goods
- The main limitation is that space, at the end of the day, is modeled in a very limited way
  - No cost of moving goods, no frictions of moving people or firms
- Other spatial models allow evaluating the importance of **adaptation** (by, e.g., migrating)
  - How many migrants should we expect because of CC?
  - How does the welfare impact of CC depend on migration constraints?

Appendix

References I

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